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The use of Bäcklund transformations in obtaining N-soliton solutions in Wronskian form

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Abstract. The Wronskian formulation of the N-soliton solutions of various nonlinear evolution equations—modified Korteweg–de Vries (with both zero and non-zero asymptotic conditions), sine-Gordon, Korteweg–de Vries, Kadomtsev–Petviashvili and Boussinesq equations—are obtained by inductive use of Bäcklund transformations in Hirota's bilinear notation.

1. Introduction

For the nonlinear evolution equations which exhibit multisoliton solutions, two important techniques have been developed by which such solutions may be obtained. The inverse scattering transform (Ablowitz and Segur 1981) has been used to solve many nonlinear equations of physical significance, and using this method one obtains the N-soliton solution in the form of some function of an $N\times N$ determinant. On the other hand, the direct method of Hirota (1980) has provided a remarkably simpler technique for obtaining the N-soliton solutions in the form of an Nth-order polynomial in N exponentials.

However, both these formulations of the N-soliton solution have the disadvantage that writing the derivatives of the solutions in an easily handled way is not possible. As a result of this, verification of these solutions by direct substitution into the evolution equations has proved to be very difficult, if not impossible. An alternative formulation of the N-soliton solutions, in terms of some function of the Wronskian determinant of N functions, exists and because differentiation of such determinants is easy and its derivatives take such a compact form, this formulation does allow verification by direct substitution. This fact has been demonstrated—together with a derivation of the N-soliton solution in the Wronskian form using the Zakharov–Shabat method (Zakharov and Shabat 1974)—for the Korteweg–de Vries (KdV) equation and its two-dimensional version the Kadomtsev–Petviashvili (KP) equation, by Freeman and Nimmo (1983a).

Bäcklund transformations (BTs) (Miura 1976, Rogers and Shadwick 1982) have also provided a means of constructing N-soliton solutions, usually by means of the associated theorem of nonlinear superposition (Lamb 1974, Barnard 1973, Hirota and Satsuma 1978). Using the BT to obtain the N-soliton by iterative means—vacuum solution to one-soliton solution to two-soliton solution etc—is very difficult if the more usual form of the solution is used, but as was demonstrated for the cylindrical KdV equation (Freeman *et al* 1981) such a procedure is made much easier by the use of the Wronskian form.

In this paper we shall use BTs of the form first introduced by Hirota (1974) to deduce, and then prove by inductive use of the BT, that the equations under consideration (KdV, KP, modified KdV (with both zero and non-zero asymptotic conditions), sine-Gordon and Boussinesq) have N-soliton solutions which have the Wronskian form. For all the above examples (with the possible exception of the modified KdV with non-zero asymptotic condition), the N-soliton solutions may also be verified by direct substitution.

It may also be shown (Freeman 1983) that the N-soliton solutions of the nonlinear Schrödinger (NLS) equation (in both attractive and repulsive cases) and of the Davey–Stewartson equation—the generalisation to two dimensions of the repulsive NLS equation—have Wronskian formulation. In the repulsive case the N-soliton takes the form of $N\times N$ Wronskian determinants. On the other hand, in the attractive case, they are the determinants of $2N\times 2N$ matrices whose four $N\times N$ submatrices each have determinants that are Wronskian: these determinants, although not themselves Wronskian, have properties with regard to differentiation which are very similar to those possessed by such determinants.

2. Properties of Wronskians

A Wronskian W is an $N\times N$ determinant of a matrix with columns $S^{(0)},\ldots,S^{(N-1)}$, where $S^{(0)}=(S_1(x,y,t),\ldots,S_N(x,y,t))^{\mathrm{T}}$ and $S^{(j)}=\partial^j S^{(0)}/\partial x^j$, written

$$W=|S^{(0)},\ldots,S^{(N-1)}|,$$

or in a more compact notation,

$$W=(\widehat{N-1}).$$

Here and below $\widehat{N-K}$ will represent the $N-K+1$ consecutive (ordered) columns $S^{(0)},\ldots,S^{(N-K)}$ and in this notation $N-K$ will indicate the single column $S^{(N-K)}$. The most useful property of a Wronskian is the compactness with which its derivatives may be written, for we have

$$\begin{aligned}\frac{\partial W}{\partial x}&=\sum_{j=0}^{N-1}\left|S^{(0)},\ldots,S^{(j-1)},\frac{\partial S^{(j)}}{\partial x},S^{(j+1)},\ldots,S^{(N-1)}\right|\\&=\sum_{j=0}^{N-1}|S^{(0)},\ldots,S^{(j-1)},S^{(j+1)},S^{(j+1)},\ldots,S^{(N-1)}|\\&=|S^{(0)},\ldots,S^{(N-2)},S^{(N)}|\\&=(\widehat{N-2},N)\end{aligned}$$

since in all other terms there are two equal columns. In a similar way higher-order x-derivatives of W may be computed to give sums of terms which depend only on the order of the derivative and not upon N.

Further, if $S^{(0)}$ is such that

$$\frac{\partial S^{(j)}}{\partial y}=\sum_{p=0}^{K}\alpha_p\frac{\partial^p S^{(j)}}{\partial x^p}$$

and

$$\frac{\partial S^{(j)}}{\partial t}=\sum_{p=0}^{L}\beta_p\frac{\partial^p S^{(j)}}{\partial x^p}$$

then the y and t derivatives of W may also be computed in a compact form. For example, suppose

$$\partial S^{(j)}/\partial y=\alpha\,\partial^2 S^{(j)}/\partial x^2;$$

then

$$\begin{aligned}\frac{\partial W}{\partial y}=\sum_{j=0}^{N-1}&|S^{(0)},\ldots,S^{(j-1)},\alpha S^{(j+2)},S^{(j+1)},\ldots,S^{(N-1)}|\\ &=\alpha[|S^{(0)},\ldots,S^{(N-3)},S^{(N)},S^{(N-1)}|+|S^{(0)},\ldots,S^{(N-2)},S^{(N+1)}|]\\ &=\alpha[-(N\hat{-}3,N-1,N)+(N\hat{-}2,N+1)]\end{aligned}$$

where we reorder the columns in the first term and hence introduce a minus sign. The final property that we shall describe makes use of the determinantal identity

$$\left(\sum_{i=1}^{N}\xi_i\right)|S^{(0)},\ldots,S^{(N-1)}|=\sum_{j=0}^{N-1}|S^{(0)},\ldots,S^{(j-1)},\xi S^{(j)},S^{(j+1)},\ldots,S^{(N-1)}|,$$

where $\xi S^{(j)}=(\xi_1\,\partial^j S_1/\partial x^j,\ldots,\xi_N\,\partial^j S_N/\partial x^j)^{\mathrm{T}}$. Suppose $S^{(0)}$ is such that for all $i=1,\ldots,N$ there is a K (the same for all i) such that

$$\partial^K S_i/\partial x^K=\xi_i S_i,$$

or more concisely

$$\xi S^{(j)}=S^{(j+K)};$$

then the above sum will reduce to the sum of only a few terms. For example, if $K=2$

$$\left(\sum_{i=1}^{N}\xi_i\right)(N\hat{-}1)=-(N\hat{-}3,N-1,N)+(N\hat{-}2,N+1).$$

We shall now give examples of how the above properties may be used to prove, by use of BTS, that the N-soliton solutions of various nonlinear evolution equations take the form of Wronskians. In each of the examples the solutions can also be verified by direct substitution into the evolution equations, as has been done elsewhere for some of the examples (Freeman and Nimmo 1983a, b, Nimmo and Freeman 1983).

3. Examples

We shall use the modified KdV as a prototype example and then show how the technique developed for that equation may be used for the other equations mentioned in the introduction.

3.1. Modified KdV *equation*

In all the examples we shall use the Hirota bilinear notation (Hirota 1980). The modified KdV equation

$$u_t+6u^2u_x+u_{xxx}=0 \tag{1}$$

has bilinear form (Hirota and Satsuma 1978)

$$(D_t + D_x^3) f \cdot f^* = 0, \tag{2a}$$

$$D_x^2 f \cdot f^* = 0, \tag{2b}$$

where $u = \mathrm{i}\, (\log f^*/f)_x$, f^* being the complex conjugate of f. A BT between two solutions f' and f of (2) is defined by the pair of relations (Hirota 1974)

$$D_x f' \cdot f^* = -\mathrm{i} k f'^* f, \tag{3a}$$

$$(D_t + 3k^2 D_x + D_x^3) f' \cdot f = 0, \tag{3b}$$

where k is an arbitrary real constant. This pair of equations is, by definition, such that if we specify f as some solution of (2), integration of (3) will generate another solution f' of (2). Thus if we take $f = 1$, corresponding to the vacuum solution $u = 0$, we obtain a pair of equations for f_1,

$$f_{1_x} = \mathrm{i} k_1 f_1^*, \tag{4a}$$

$$f_{1_t} + 3k_1^2 f_{1_x} + f_{1_{xxx}} = 0. \tag{4b}$$

Eliminating the complex conjugate from (4*a*) we obtain

$$f_{1_{xx}} = k_1^2 f_1. \tag{4c}$$

The general solution of (4) is

$$f_1 = (1-\mathrm{i})\alpha_1 \exp(k_1 x + 4k_1^3 t) + (1+\mathrm{i})\beta_1 \exp(-k_1 x + 4k_1^3 t), \tag{5}$$

where α_i and β_i are arbitrary real constants. If we now use the solution f_1 given by (5) to generate another solution f_2 and so on, we will generate a hierarchy of solutions f_1 to f_N. We postulate that f_N is given by

$$f_N = (\widehat{N-1}), \tag{6}$$

using the notation already defined, where

$$S_i = (1-\mathrm{i})\alpha_i \exp(k_i x - 4k_i^3 t) + (1+\mathrm{i})\beta_i \exp(-k_i x + 4k_i^3 t), \tag{7}$$

for $i = 1, \ldots, N$, where α_i, β_i and k_i are arbitrary real constants. This form of the N-soliton solution has been obtained by independent means by Satsuma (1979).

In fact what we postulate is that the N-soliton solution of (1) is given by

$$f_N = (\widehat{N-1}) = |S^{(0)}, \ldots, S^{(N-1)}|,$$

where $S^{(0)}$ is such that

$$k^2 S^{(j)} = S^{(j+2)} \qquad \text{and} \qquad S_t^{(j)} = -4S^{(j+3)},$$

using (4), and where

$$k^2 S^{(j)} = (k_1^2\, \partial^j S_1/\partial x^j, \ldots, k_N^2\, \partial^j S_N/\partial x^j).$$

In order to verify this solution we shall show that the BT (3) transform between

$$f_{N-1} = (\widehat{N-2}) \qquad \text{and} \qquad f_N = (\widehat{N-1}).$$

Computing the various derivatives of f_{N-1} and f_N in accordance with the procedure described in § 2, we have for f_N

$$f_{N_x} = (\widehat{N-2}, N), \qquad f_{N_{xx}} = (\widehat{N-3}, N-1, N) + (\widehat{N-2}, N+1),$$

$$f_{N_{xxx}} = (N\hat{-}4, N-2, N-1, N) + 2(N\hat{-}3, N-1, N+1) + (N\hat{-}2, N+2),$$

$$f_{N_t} = -4[(N\hat{-}4, N-2, N-1, N) - (N\hat{-}3, N-1, N+1) + (N\hat{-}2, N+2)],$$

and similar expressions for the derivatives of f_{N-1}. As well as these derivatives we need to determine f_N^*, and this may be achieved by noting that the S_i satisfy

$$\partial^j S_i^* / \partial x^j = \mathrm{i}k_i\, \partial^{j-1} S_i / \partial x^{j-1},$$

and hence $f_N^* = \Pi_{i=1}^N (\mathrm{i}k_i)(-1, N\hat{-}2)$. Substitution of these expressions into (3a) now yields

$$[(-1, N\hat{-}3, N-1)(N\hat{-}2) - (-1, N\hat{-}2)(N\hat{-}3, N-1)] \prod_{i=1}^{N} \mathrm{i}k_i$$

$$-\mathrm{i}k_N \left((N\hat{-}1)(-1, N\hat{-}3) \prod_{i=1}^{N-1} \mathrm{i}k_i \right)$$

which is the expansion by $N \times N$ minors of the $(2N-1) \times (2N-1)$ determinant

$$\prod_{i=1}^{N} \mathrm{i}k_i \begin{vmatrix} N\hat{-}3 & \cdot & -1 & N-2 & N-1 \\ \cdot & N\hat{-}3 & -1 & N-2 & N-1 \end{vmatrix} \begin{array}{l} N \text{ rows} \\ N-1 \text{ rows} \end{array}$$

which can easily be shown to be equal to zero by means of elementary row and column operations. Note that in order for this determinant to vanish we must have, for $i = 1, \ldots, N-1$, the constants α_i, β_i and k_i in f_N equal to the corresponding constants in f_{N-1}.

In a very similar way equation (3b) may be shown to give

$$\begin{aligned} -6[&(N\hat{-}4, N-2, N-1, N)(N\hat{-}2) + (N\hat{-}4, N-2, N)(N\hat{-}1) \\ &- (N\hat{-}4, N-2, N-1)(N\hat{-}2, N) - (N\hat{-}3, N-1, N+1)(N\hat{-}2) \\ &- (N\hat{-}3, N+1)(N\hat{-}1) + (N\hat{-}2, N+1)(N\hat{-}3, N-1)] \end{aligned}$$

having made use of the two identities

$$\left(\sum_{i=1}^{N} k_i^2 \right)(N\hat{-}1) = -(N\hat{-}3, N-1, N) + (N\hat{-}2, N+1),$$

$$\left(\sum_{i=1}^{N} k_i^2 \right)(N\hat{-}2) = -(N\hat{-}4, N-2, N-1) + (N\hat{-}3, N),$$

of the kind mentioned in § 2, and this may be written as

$$-6 \begin{vmatrix} N\hat{-}4, N-2 & \cdot & N-3 & N-1 & N \\ \cdot & N\hat{-}4, N-2 & N-3 & N-1 & N \end{vmatrix}$$

$$+6 \begin{vmatrix} N\hat{-}3 & \cdot & N-2 & N-1 & N \\ \cdot & N\hat{-}3 & N-2 & N-1 & N \end{vmatrix}$$

which are zero as for (3a). So, by induction, we have proved that the Nth integral of the BT (3) is $f_N = (N\hat{-}1)$.

This solution is real for all real α_i, β_i and k_i and is thus the N-soliton solution of the modified KdV equation. In other examples, some of which we shall consider here, the soliton solutions will form only a (real) subset of the (complex) solutions obtained from the BT.

3.2. *Modified KdV with non-zero asymptotic condition*

This equation

$$u_t + 6u^2 u_x + u_{xxx} = 0, \qquad u \to u_0 \text{ as } |x| \to \infty, \tag{8}$$

is equivalent to the mixed KdV (MKdV) equation

$$u_t + 6u^2 u_x + 6uu_x + u_{xxx} = 0, \qquad u \to 0 \text{ as } |x| \to 0,$$

and has been considered by Ono (1976) and in the second form by Wadati (1975). Equation (8) has bilinear form (Ablowitz and Satsuma 1978)

$$(D_t + 6u_0^2 D_x + D_x^3) f^* \cdot f = 0, \tag{9a}$$

$$(D_x^2 - 2\mathrm{i}u_0 D_x) f^* \cdot f = 0, \tag{9b}$$

where $u = u_0 + \mathrm{i}[\log(f^*/f)]$, and has BT

$$D_x f' \cdot f^* = -\mathrm{i}(u_0 f' f^* + k f f'^*), \tag{10a}$$

$$\{D_t + [6u_0^2 + 3(k^2 - u_0^2)] D_x + D_x^3\} f' \cdot f = 0. \tag{10b}$$

Guided by the above example we put $f = 1$ corresponding to the vacuum solution $u = u_0$ and hence obtain from (10)

$$f_{1_x} = -\mathrm{i}(u_0 f_1 + k_1 f_1^*), \tag{11a}$$

$$f_{1_{xx}} = (k_1^2 - u_0^2) f_1, \tag{11b}$$

$$f_{1_t} + [6u_0^2 + 3(k_1^2 - u_0^2)] f_{1_x} + f_{1_{xxx}} = 0, \tag{11c}$$

so that we can take as the Nth solution

$$f = (N \hat{-} 1),$$

where

$$S_i = \alpha_i \exp[l_i x - (6u_0^2 + 4l_i^2) l_i t] + \beta_i \exp[-l_i x + (6u_0^2 + 4l_i^2) l_i t]$$

and where $l_i = +(k_i^2 - u_0^2)^{1/2}$, and from (11$a$) we have

$$\alpha_i^*/\alpha_i = (-u_0 + \mathrm{i}l_i)/k_i \qquad \text{and} \qquad \beta_i^*/\beta_i = (-u - \mathrm{i}l_i)/k_i.$$

As a result of these latter conditions we see that

$$S_i^* = (1/k_i)(-u_0 S_i + \mathrm{i}S_{i_x}),$$

and consequently,

$$\begin{aligned} f_N^* &= \prod_{i=1}^{N} \frac{1}{k_i} |-u_0 S^{(0)} + \mathrm{i}S^{(1)}, \ldots, -u_0 S^{(N-1)} + \mathrm{i}S^{(N)}| \\ &= \left(\prod_{i=1}^{N} \frac{1}{k_i} \right) \sum_{j=0}^{N} (-u_0)^j \mathrm{i}^{N-j} (\hat{N})_j \end{aligned}$$

where $(\hat{N})_j$ is the determinant of the matrix with columns $S^{(0)}, \ldots, S^{(j-1)}$, $S^{(j+1)}, \ldots, S^{(N)}$. The fact that f_N^* is a sum rather than just a single term makes the proof that the BT transform between f_{N-1} and f_N by direct substitution a little more difficult; however, it is still possible. We compute $f^*_{N_x}$ and regroup terms to give

$$f^*_{N_x}=\Pi_N\sum_{j=0}^{N-1}[(-u_0)^{j+1}\mathrm{i}^{N-j-1}(\hat{N})_j+(-u_0)^j\mathrm{i}^{N-j}(N\hat{-}1,N+1)_j],$$

where $\Pi_N=\Pi_{i=1}^N(1/k_i)$.

Equation ($10a$) now gives the terms

$$\Bigg[(N\hat{-}2,N)\sum_{j=0}^{N-1}(-u_0)^j\mathrm{i}^{N-j-1}(N\hat{-}1)_j$$
$$-(N\hat{-}1)\sum_{j=0}^{N-2}((-u_0)^{j+1}\mathrm{i}^{N-j-2}(N\hat{-}1)_j+(-u_0)^j\mathrm{i}^{N-j-1}(N\hat{-}2,N)_j)$$
$$+(N\hat{-}1)\sum_{j=0}^{N-1}(-u_0)^{j+1}\mathrm{i}^{N-j-2}(N\hat{-}1)_j$$
$$+(N\hat{-}2)\sum_{j=0}^{N}(-u_0)^j\mathrm{i}^{N-j-1}(\hat{N})_j\Bigg]\Pi_{N-1}.$$

The terms in the summation with $j>N-2$ vanish identically and we are left with

$$\Pi_{N-1}\sum_{j=0}^{N-1}(-u_0)^j\mathrm{i}^{N-j-1}[(N\hat{-}2,N)(N\hat{-}1)_j-(N\hat{-}1)(N\hat{-}2,N)_j-(N\hat{-}2)(\hat{N})_j],$$

which is the Laplace expansion (by $N\times N$ minors) of the $(2N-1)\times(2N-1)$ determinant

$$\Pi_{N-1}\sum_{j=0}^{N-2}(-u_0)^j\mathrm{i}^{N-j-1}\begin{vmatrix}(N\hat{-}2)_j & & j & N-1 & N\\ & (N\hat{-}2)_j & j & N-1 & N\end{vmatrix}$$

which is equal to zero in the usual way. Verification that ($10b$) is satisfied is much easier since it does not involve complex conjugates and hence does not contain summations, and thus the BT is proved.

3.3. *Sine–Gordon equation*

The details for this equation are very similar to those for the MKdV equation of § 3.1. The sine-Gordon equation

$$u_{xt}=\sin u \tag{12}$$

has bilinear form

$$D_xD_tf\cdot f=\tfrac{1}{2}(f^2-f^{*2}) \tag{13}$$

together with its complex conjugate, where $u=\mathrm{i}(\log f^*/f)$ (Hirota and Satsuma 1978), which has the BT

$$D_xf'\cdot f^*=kf'^*f, \tag{14a}$$

$$D_tf'\cdot f=(1/4k)f'^*f^*, \tag{14b}$$

which is slightly different from the BT obtained by Hirota (1974) and is obtained by considering f' satisfying (13) and f satisfying the complex conjugate. By taking $f=1$

we get

$$f_{1_x} = k_1 f_1^*, \qquad f_{1xx} = k_1^2 f_1, \tag{15a, b}$$

$$f_{1_t} = (4k_1)^{-1} f_1^*, \qquad f_{1_{tt}} = (16k_1^2)^{-1} f_1, \tag{15c, d}$$

so that we may deduce that

$$f_N = (\widehat{N-1}),$$

where

$$S_i = \alpha_i \exp\left(k_i x + \frac{1}{4k_i} t\right) + \mathrm{i}\beta_i \exp\left(-k_i x \frac{1}{4k_i} t\right)$$

with α_i, β_i and k_i all real.

Also it is easy to show that (cf § 3.1)

$$f_N^* = \prod_{i=1}^{N} k_i(-1, \widehat{N-2}) = \prod_{i=1}^{N} \frac{1}{k_i} (\tilde{N})$$

where $(\tilde{N}) \equiv |S^{(1)}, \ldots, S^{(N)}|$. Verifying that (14) transforms between $f_{N-1} = (\widehat{N-2})$ and f_N is now straightforward.

3.4. KP, KdV and Boussinesq equations

Finally, we shall consider some examples that we have considered from a different viewpoint elsewhere (Freeman and Nimmo 1983a, b, Nimmo and Freeman 1983).

The KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

has bilinear form

$$(D_x D_t + D_x^4 + 3D_y^2) f \cdot f = 0, \tag{16}$$

where $u = 2(\log f)_{xx}$, and BT (Hirota 1980)

$$(D_x^2 - D_y) f' \cdot f = 0, \tag{17a}$$

$$(D_t - 3D_x D_y + D_x^2) f' \cdot f = 0. \tag{17b}$$

Again, if we take $f = 1$ corresponding to the vacuum solution we obtain from (17) a solution f' corresponding to the two-soliton solution from which we deduce the form of the N-soliton solution as

$$f_N = (\widehat{N-1}),$$

where

$$S_i = \alpha_i \exp(l_i x + l_i^2 y - 4l_i^3 t) + \beta_i \exp(-n_i x + n_i^2 y + 4n_i^3 t).$$

This solution may then be verified using (17) in the usual way (see Freeman and Nimmo 1983a, b). To obtain the N-soliton solution of the KdV equation we take $n_i = l_i = k_i$, in which case a factor of $\exp(k_i^2 y)$ may be taken out of each row of the Wronskian and upon taking $u = 2(\log f)_{xx}$ these factors make no contribution to u. In order to get the (regular) soliton solutions we take the k_i's to be real and $\beta_i/\alpha_i = (-1)^{i+1}$ for $i = 1, \ldots, N$ where the k_i's are ordered so that $k_1 < k_2 < \ldots < k_N$.

Similarly for the Boussinesq equation

$$u_{tt}-u_{xx}-6(u^2)_{xx}-u_{xxxx}=0,$$

in bilinear form,

$$(D_t^2-D_x^2-D_x^4)f\cdot f=0, \tag{18}$$

where $u=(\log f)_{xx}$ (Hirota 1973) and BT (Hirota and Satsuma 1977, Nimmo and Freeman 1983)

$$(D_t+\mathrm{i}\sqrt{3}\,D_x^2)f'\cdot f=0, \tag{19a}$$

$$(\mathrm{i}\sqrt{3}\,D_xD_t+D_x+D_x^3)f'\cdot f=kf'f. \tag{19b}$$

From this we obtain

$$f=(\widehat{N-1}),$$

where

$$S_i=\alpha_i\exp(l_ix-\mathrm{i}\sqrt{3}\,l_i^2t)+\beta_i\exp(m_ix-\mathrm{i}\sqrt{3}\,m_i^2t)+\gamma_i\exp(\eta_ix-\mathrm{i}\sqrt{3}\,n_i^2t)$$

and l_i, m_i and n_i are the three roots of $4p_i^3+p_i=k_i$.

4. Conclusions

A method of using BTS to obtain N-soliton solutions has been described. This method allows one to verify solutions directly and to prove by induction that the Nth solution of the hierarchy of integrals of the BT is a Wronskian of N functions which differ from one another only parametrically. The Wronskian formulation has also been extended recently to the soliton solutions of other equations—nonlinear Schrödinger equations (Freeman 1983) and partial differential–difference equations (Nimmo 1983)—however, the scope of this approach is still not yet fully understood.

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